2. POLUBARINOVA-KOCHINA P.YA., On the filtration in anisotropic soil. PMM, 4, 2, 1940.
3. BRAGINSKAYA V.A., Some problems of filtration in anisotropic soil. PMM, 6, 2-3, 1942.
4. PAVLOV A.T., Steady flow of ground waters with two layers of liquid of different density. PMM, 6, 2-3, 1942.
5. POLUBARINOVA-KOCHINA P.YA., Some Problems of Plane Flows of Ground Waters. Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1942.
6. POLUBARINOVA-KOCHINA P.YA., Theory of the Motion of Ground Waters. Nauka, Moscow, 1977.
7. BERESLAVSKII E.N., On the conformal mapping of certain circular polygons onto a rectangle. Izv. Vuz. Matematika, 5, 1980.
8. BATEMAN H. and ERDELYI A., Higher Transcendental Functions. McGraw-Hill, New York, 1955.

Translated by L.K.

PMM U.S.S.R.,Vol.54,No.2,pp.289-292,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

ESTIMATES OF THE PARAMETERS OF INCREASING PERTURBATIONS IN SHEAR FLOWS OF AN INHOMOGENEOUS MAGNETIZED PLASMA*

N.A. GLAZUNOVA and YU.A. STEPANYANTS

The method of integral relations is used to obtain estimates of the phase velocity and perturbation growth increment in the shear flow of a magnetized plasma, analogous to existing estimates /1, 2/ in the hydrodynamics of stratified fluid, and to refine the results obtained in /3/.

1. We shall start with a well-known system of equations of magnetohydrodynamics for an ideal incompressible fluid of variable density in a gravitational force field /4/:

$$\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = \frac{1}{\rho+\rho_1}\left\{-\nabla p + g\rho_1 - \frac{1}{4\pi}[\mathbf{B}\times \operatorname{rot}\mathbf{B}]\right\} \tag{1.1}$$
$$\partial_t \mathbf{B} = \operatorname{rot}[\mathbf{v}\times\mathbf{B}], \ \operatorname{div}\mathbf{B} = 0$$
$$\partial_t \rho_1 + (\mathbf{v}\nabla)(\rho+\rho_1) = 0, \ \operatorname{div}\mathbf{v} = 0$$

Here p and ρ_1 are the pressure and density perturbations, $\rho(z)$ is the unperturbed density distribution along the vertical, and the remaining notation is traditional.

Let the fluid be contained between two horizontal solid boundaries $z=0$ and $z=H$. The components of the flow velocity vector and magnetic field strength have, in the unperturbed state, the form $\{U(z), 0, 0\}$, $\{B_0(z), 0, 0\}$. We shall assume that the perturbations of these fields are two-dimensional: $\mathbf{v}=\{u, 0, w\}$, $b=\{b_x, 0, b_z\}$. Linearizing the initial system of equations and seeking the solutions in the form of a product obtained by multiplying the corresponding structural functions depending on z by $\exp\{ik(x-ct)\}$, we reduce the system (1.1) to a single equation for the auxiliary function $f(z)=F(z)[U(z)-c]^n/\rho'$ (a prime denotes a derivative with respect to z), where $F(z)$ is a function defining the structure of the density perturbation along the vertical. We multiply the equation obtained in this manner by a complex conjugate function $\bar{f}(z)$, and integrate the result in z from 0 to H. As a result we arrive at the following integral relation (from now on the limits of integration will be ommitted for simplicity):

$$\begin{gathered}\int \rho[(U-c)^{2(1-n)} - V_A^2(U-c)^{-2n}][|f'|^2 + k^2|f|^2]\,dz + \\ \int\{\rho(U-c)^{-2n}[n(1-n)U'^2 - N^2] + n(U-c)^{1-2n}(\rho U')' - \\ n(U-c)^{-2n-1}U'(\rho V_A^2)' - n\rho V_A^2(U-c)^{-2n-1}U'' + \\ n(n+1)\rho V_A^2(V-c)^{-2(n+1)}U'^2\}|f|^2dz = 0\end{gathered} \tag{1.2}$$

*Prikl.Matem.Mekhan.,54,2,346-349,1990

Here n is an arbitrary number, $N^2(z) = -g\rho'/\rho$ is the square of the frequency of buoyancy, $V_A^2 = B_0^2/(4\pi\rho)$ is the square of the Al'fen velocity and c is the phase velocity of the perturbations, which represents for the given real value of the wave number k, a complex spectral parameter of the boundary-value problem for the equation mentioned above, with the boundary conditions $f(0) = f(H) = 0$.

Eq.(1.2) represents, essentially, a set of independent integral relations for various values of the parameter n. Everyone of these relations can be used to derive some estimate for the spectral parameter c. In the usual hydrodynamics of a stratified liquid, the passage to which takes place naturally when $B_0, V_A^2 \to 0$, various estimates for c were successfully obtained for the cases $n = 0, 1/2, 1$ /1, 2, 5–9/. In magnetohydrodynamics the situation is more complicated, and only the case $n = 0$ has been investigated.

2. Let us consider the case of $n = 0$. We shall assume that the basic flow is unstable, so that the phase velocity of the perturbations is complex $c = c_r + ic_i$, while $c_i > 0$. The imaginary part of the integral Eq.(1.2) has the following form at $n = 0$:

$$\int \rho (U - c_r) Q dz = 0, \quad Q = |f'|^2 + k^2 |f|^2$$

Let us write the real part of the corresponding integral relation in the form

$$\int \rho U^2 Q dz = I_1 + I_2 + |c|^2 I$$
$$I_1 = \int \rho V_A^2 Q dz, \quad I_2 = \int \rho N^2 |f|^2 dz, \quad I = \int \rho Q dz$$

Let the velocity profile be bounded and contained within the interval $[0, H]$, within the range $U_{\min} \leqslant U(z) \leqslant U_{\max}$. We shall use the inequality

$$0 \geqslant \int \rho (U - U_{\min})(U - U_{\max}) Q dz = I_1 + I_2 + \chi I \geqslant \tag{2.1}$$
$$(\chi + V_{A\min}^2) I + I_2$$
$$\chi = (c_r - U_+)^2 + c_i^2 - U_-^2, \quad U_\pm = (U_{\max} \pm U_{\min})/2$$

If we reject from this inequality the integral I_2 which is already known to be positive (under the assumption that the stratification is statically stable, i.e. $N^2(z) > 0$), then the inequality will only become stronger. In this case the inequality will yield the result of /10/, i.e. the "semicircle" theorem which states that the complex phase velocity of increasing perturbations is contained within a semicirle in the complex plane with centre at the point $(U_+, 0)$ of radius $R = \sqrt{U_-^2 - V_{A\min}^2}$ (the semicircle is shown in the figure by a dahsed line). However, dropping the integral I_2 containing N^2 means, actually, that the stratification is neglected (the liquid is of uniform density). Kochar and Jain /3/ were successful in taking into account the influence of this integral by expressing it in terms of the integral I, but the process involved the use of fairly rough estimates, and this led to a final result which did not include the dependence of the range of possible values of c on the wave number k. Below we give more accurate estimates analogous to those used in /1, 2/.

3. We shall use the auxiliary integral relations for the function $G = f\sqrt{U - c}$. The relation is obtained after carrying out simple but bulky transformations analogous to those used in deriving relations (1.2). The imaginary part of the new integral relation yields ($N^2/U'^2 \equiv J(z)$ is Richardson's number):

$$\int \rho (1 + V_A^2 |U - c|^{-2})(|G'|^2 + k^2 |G|^2) dz - \tag{3.1}$$
$$\int \rho \{(1/4 - J) U'^2 - (U - c_r)|U - c|^{-2}[V_A^2 U'' +$$
$$U'(\rho V_A^2)/\rho] + 3/4\, V_A^2 |U - c|^{-4} U'^2 [3(U - c_r)^2 - c_i^2]\} |U -$$
$$c|^{-2} |G|^2 dz = 0$$

Dropping from (3.1) the term known to be positive and containing the cofactors $V_A^2 |U - c|^{-2}$, and estimating the lower limit of the remaining integrals, we obtain the inequality

$$\Lambda B^2 \geqslant D^2; \quad \Lambda = 1/4 - J_{\min} + (\mu^2 + \nu^2)/c_i^2 \tag{3.2}$$

$$B^2 = \int \rho U'^2 |U - c|^{-1} |G|^2 dz, \quad D^2 = \int \rho (|G'|^2 +$$
$$k^2 |G|^2) dz$$
$$\mu^2 = 9/4\, V_{A\max}^2 + \max |V_A^2 (c_r - U) U'' / U'^2|$$
$$\nu^2 = \max |(\rho V_A^2)'(c_r - U)/(\rho U')|$$

Using the inequality

$$|G'|^2 \geqslant |U - c| |f'|^2 + 1/4\, U'^2 |f|^2 / |U - c| - |U'| |f'| |f|$$

and the Cauchy-Bunyakovskii-Schwartz inequality to obtain the estimate of the integral

$$\int \rho U' |f'| |f| dz \leqslant AB, \quad A^2 = \int \rho |U - c| |f'|^2 dz$$

we obtain the following estimate for the right-hand side of (3.2):

$$D^2 \geqslant A^2 + {}^1/_4 B^2 + k^2C^2 - AB, \quad C^2 = \int \rho |U - c| |f|^2 dz \tag{3.3}$$

An inequality analogous to (3.3) was obtained in /3, 6/, but the terms A^2 and k^2C^2 were combined in it to form a single term. This led to the fact that after carrying out the subsequent estimates the result because independent of the wave number k, i.e. a rough estimate was obtained, uniform in k. However, as was shown in /1, 7/, everyone of the integrals A, B, C could be estimated separately, and this is what we have done below.

Combining the inequalities (3.2) and (3.3), we write

$$(A / B - {}^1/_2)^2 \leqslant \Lambda - k^2C^2 / B^2 \tag{3.4}$$

The definitions of B and C yield the estimate $C^2/B^2 \geqslant c_i^2/U'^2_{max}$, for the ratio C^2/B^2. Taking this into account, we can write the inequality (3.4) in the form

$$A^2 + k^2C^2 \leqslant MB^2, \quad M = {}^1/_4 + \Lambda + \sqrt{\Lambda - k^2c_i^2/U'^2_{max}} \tag{3.5}$$

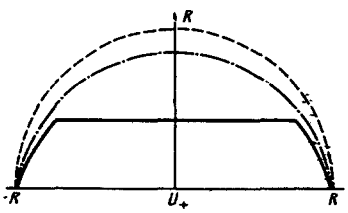
On the other hand, the definitions of the integrals A, B, C yield

$$A^2 + k^2C^2 \geqslant c_i I, \quad B^2 \leqslant I_3/c_i, \quad I_3 = \int \rho U'^2 |f|^2 dz$$

Substituting these estimates into the inequality (3.5) we obtain $I_3 \geqslant c_i^2 I/M$. Moreover, we have $I_2 \geqslant J_{min} I_3$. Using the last two inequalities, we finally obtain from (2.1):

$$(c_r - U_+)^2 + c_i^2 (1 + J_{min}/M) \leqslant R^2 \tag{3.6}$$

The latter inequality determines, in the complex c-plane, the domain of possible values of the phase velocity, and the form of the boundary curve depends here explicitly on the minimum value of Richardson's number J_{min}, as well as on the wave number k of the perturbation.



Putting $\mu = \nu = 0$, in (3.6) we obtain, for infinitely long perturbations $(k \to 0)$, the result of /3/. The boundary curve becomes in this case a semi-ellipse whose semi-minor axis in the direction of c_i depends explicitly on J_{min} (the dot-dash curve in the figure). In the limit, as $J_{min} \to 0$, the semi-ellipse becomes a semicircle /10/.

It can be shown that in the general case $k \neq 0$, $J_{min} \neq 0$ the curve serving as the boundary of the region (3.6) will always lie within the semicircle /10/ and also within the semi-ellipse /3/ (the solid line in the figure). When there is no external magnetic field, when $V_A^2 = 0$, we obtain from (3.6) the already known result for the usual hydrodynamics of a stratified fluid /1, 7/.

We shall mention another inequality which follows from (3.6), representing the requirement that the radicand in the definition of M (see (3.5)) should be non-negative:

$$2k^2c_i^2 \leqslant ({}^1/_4 - J_{min}) U'^2_{max} + |U'|_{max} \sqrt{({}^1/_4 - J_{min})^2 U'^2_{max} + 4k^2(\mu^2 + \nu^2)} \tag{3.7}$$

The product kc_i represents the increment in the growth of perturbations, therefore the upper limit of inequality (3.7) can be estimated.

Finally we note that when the magnetic field strength increases (i.e. when $V_{A\,min}$ increases), we see from the inequality (3.6) that the size of the region containing the complex quantity c decreases. When the field strength B_0, is sufficiently large and such that $V^2_{A\,min} > U_-^2$, the right-hand side of inequality (3.6) becomes negative and the inequality itself loses its meaning. This corresponds to the absence of instability in the shear flow, since we have, within the framework of ideal magnetohydrodynamics without dissipation discussed here, the "freezing in" of the lines of force /4/, which represses the oscillations across the external magnetic field. In this sense, the magnetic field exerts a stabilizing influence on the shear flow, analogous to the surface tension between two mutually immiscible liquids.

REFERENCES

1. MAKOV YU.N. and STEPANYANTS YU.A., On the parameters of increasing waves in shear flows. Okeanologiya. 23, 3, 1983.
2. MAKOV YU.N. and STEPANYANTS YU.A., On the effect of the curvature of the velocity profile on the increasing waves in shear flows. Okeanologiya., 24, 4, 1984.
3. KOCHAR G.T. and JAIN R.K., On Howard's semicircle theorem in hydromagnetics. J. Phys. Soc. Japan. 47, 2, 1979.
4. LANDAU L.D. and LIFSHITZ E.M., Theoretical Physics. 8, Electrodynamics of Continuous Media. Nauka, Moscow, 1982.
5. HOWARD L.N., Note on a paper of John W. Miles. J. Fluid Mech. 10, 4, 1961.
6. KOCHAR G.T. and JAIN R.K., Note on Howard's semicircle theorem. J. Fluid Mech. 91, 3, 1979.
7. MAKOV YU.N. and STEPANYANTS YU.A., Note on the paper of Kochar and Jain on Howard's semicircle theorem. J. Fluid Mech. 140, 1984.
8. MAKOV YU.N. and STEPANYANTS YU.A., On the effect of stratification on the stability of shear flows of an ideal fluid. Dokl. Akad. Nauk SSSR, 284, 5, 1985.
9. MILES J.W., On the stability of heterogeneous shear flows. J. Fluid Mech. 10, 4, 1961.
10. AGRAWAL S.C. and AGRAWAL G.S., Hydromagnetic stability of heterogeneous shear flow. J. Phys. Soc. Japan. 27, 1, 1969.

Translated by L.K.